ell. cusp'tion $\rightsquigarrow$ • mono-theta env.

const. mult. rig. $\rightsquigarrow$ non-interference; can. splitting

• recon. in Arch. theory (Aut-hol.)

• recon of NF (gl. non-realif'd) & Kummer theory for gl. portion

Belyi cusp'tion

cycl. rig.

$F^{\times} \cap \prod_{v} \mathcal{O}_v = \mu$ $\rightsquigarrow$ non-interfer-ce for "gl portion"

ring str. { $\boxtimes$-line bdl; $\boxplus$-line bdl

concerning "val. gp portion" of Θ-link

we can protect val. gp portion from $\hat{\mathbb{Z}}^{\times}$-indet.

Θ-ll with $\dagger\mathcal{O}^{\times} \cong \ddagger\mathcal{O}^{\times}$

$\hat{\mathbb{Z}}^{\times}$-indet

• val. gp core $\{\underline{\underline{q}}^{j^2}\} \mapsto \underline{\underline{q}}$

• gl. real'd $\mathbb{R}_{\geq 0}$

Prop 3.14 (cycl. rig. for inertia subgps [AbsTopIII, Prop 1.4])

X gen ≥ 2 $\subset \overline{X}$

$\phi \neq U \underset{\text{open}}{\subset} X$ $x \in X(k) \setminus U(k)$ $U_x := \overline{X} \setminus \{x\}$

I_x : inertia at x $\subset \Delta_U$

(1). $\ker(\Delta_U \twoheadrightarrow \Delta_{U_x})$, $\ker(\Pi_U \twoheadrightarrow \Pi_{U_x})$ top. normally gen. by the inertia subgps of the pts of $U_x \setminus U$

(2). $1 \to I_x \to \Delta_{U_x}^{\text{cusp-cent}} \to \Delta_{\overline{X}} \to 1$

$\underset{\text{Leray}}{\leadsto}$ $E_2^{p,q} = H^p(\Delta_{\overline{X}}, H^q(I_x, I_x)) \Rightarrow H^{p+q}(\Delta_{U_x}^{\text{cusp-cent}}, I_x)$

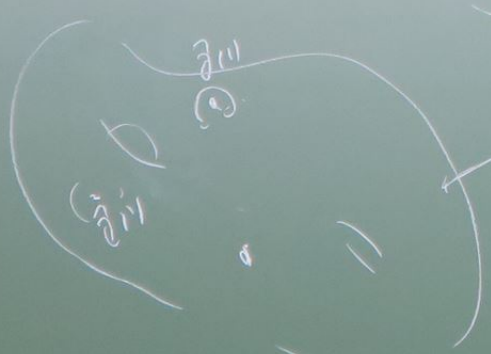
$I_x, \Delta_{U_x}^{\text{cusp-cent}} \curvearrowright I_x$ by conj.

mono-theta env. non-interference

$$\sim \quad \hat{\mathbb{Z}} = H_{om}(I_x, I_x) \cong H^0(\Delta_{\bar{X}}, H^1(I_x, I_x)) = E_2^{0,1}$$

$$\longrightarrow E_2^{2,0} = H^2(\Delta_{\bar{X}}, H^0(I_x, I_x))$$

$$\cong H_{om}(\mu_{\hat{\mathbb{Z}}}(\Pi_X), I_x)$$



orient. $\hat{\mathbb{Z}}(1)$

$G_{\mathbb{Q}} \subset \widehat{GT}$

(Cyc. Rig. Iner.) $\mu_{\hat{\mathbb{Z}}}(\Pi_X) \xrightarrow{\sim} I_x$

cyclotomic rigidity for inertia subgp

$\hat{\mathbb{Z}}^\times$

$H^2 \cong \hat{\mathbb{Z}}(-1)$

proof) omit //

Lem 3.15 ([AbsTopIII, Prop 1.6])

k: Kummer-faithful, X: proper

$\phi \neq U \underset{\text{open}}{\subset} X$

$$\kappa_U : \Gamma(U, \mathcal{O}_U^\times) \to H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\overline{k(X)})) = H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\bar{k}))$$

$$\cong H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$$

$\uparrow$ $\mu_{\hat{\mathbb{Z}}}(\bar{k}) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$

(1). κ_U : inj

$\setminus U$

$, I_K)$

$\mu_{\widehat{\ell}}(\bar{k}))$

$\ldots, \mu_{\widehat{\ell}}(\Pi_X))$

$\cong \mu_{\widehat{\ell}}(\Pi_X))$

(2). (cf. [Cusp, Prop 2.3 (ii)])

$\forall$ divisor D of $\deg = 0$ on X s.t. $\mathrm{Supp}(D) \subset X(k)$,

the section $s_D : G_k \to \Pi_J$ is equal to

(up to conj by Δ_X) the section

det'd by the origin $O \in J(k)$

$\Leftarrow$) D is principal

X/k

J^d $\deg = d$

$J = J^0$ ...

$X \to J^1$

$P \mapsto \mathcal{O}(P)$

$\Pi_X \to \Pi_{J^1}$

$x \in X(k)$

$s_x : G_k \to \Pi_X \to \Pi_{J^1}$

$\Pi_{J^1} \times \cdots \times \Pi_{J^1} \to \Pi_{J^d}$

div D $\deg = d$ s.t. $\mathrm{Supp}(D) \subset X(k)$

$\rightsquigarrow s_D : G_k \to \Pi_{J^d}$

(3). (cf. [Cusp, Prop 2.1 (ii)]

$U = X \setminus S$, $S \subset X(k)$ fin. set

$\Pi_U \twoheadrightarrow \Pi_U^{\text{cusp-cent}}$

$\underset{\text{induces}}{\sim}$ $H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(4). (cf. [Cusp, Prop 1.4 (iii)]

$H^1(\Pi_X, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \cong (k^\times)^\wedge$

(5). (cf. [Cusp, Prop 2.1 (ii)]

restrictions to I_x $(x \in S)$

(Cusp, Prop, Iner.)

$\hat{\mathbb{Z}}$

$\wr\Vert$

$\mathrm{Hom}_{\Pi_X}(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$0 \to H^1(\Pi_X, H^0(\prod_{x \in S} I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} H^0(\Pi_X, H^1(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X)))$

(2). (cf. [Cusp, Prop 2.3 (ii)]

$X \setminus k$

$\mathrm{Supp}(D) \subset X(k)$

(1). $\mathrm{Ker}(\Delta_U \to \Delta_{U_x})$, $\mathrm{Ker}(\Pi_U \to \Pi_{U_x})$ top. normally gen. by the inertia gps of the pts of $U_x \setminus U$

(2). $1 \to I_x \to \Delta_{U_x}^{\text{cusp-cent}} \to \Delta_{\bar{X}} \to 1$

$$\underset{\text{Leray}}{\rightsquigarrow} \quad E_2^{p,q} = H^p(\Delta_{\bar{X}}, H^q(I_x, I_x)) \Rightarrow H^{p+q}(\Delta_{U_x}^{\text{cusp-cent}}, I_x)$$

$I_x, \Delta_{U_x}^{\text{cusp-cent}} \curvearrowright I_x$ by conj.

$$\underset{(3),(4)}{\rightsquigarrow} \quad 1 \to (k^\times)^\wedge \to H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x\in S} \hat{\mathbb{Z}}$$

(6). the image of $\Gamma(U, \mathcal{O}_U^\times)$ in $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))/(k^\times)^\wedge$ via κ_U is equal to the inverse image in $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))/(k^\times)^\wedge$ of the submodule $P'_X \subset \bigoplus_{x\in S} \mathbb{Z} \; (\subset \bigoplus_{x \in S} \hat{\mathbb{Z}})$ det'd by the principal divisors w/ support in S.

restrictions to I_x $(x \in S)$

$$0 \to H^1(\Pi_X, H^0(\prod_{x\in S} I_x, \mu_{\hat{\ell}}(\Pi_X))) \to H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\ell}}(\Pi_X)) \to \bigoplus_{x \in S} H^0(\Pi_X, H^1(I_x, \mu_{\hat{\ell}}(\Pi_X)))$$

$\mathrm{Hom}_{\Pi_X}(I_x, \mu_{\hat{\ell}}(\Pi_X))$

$, H^0(I_x, I_x))$

$\mathrm{Hom}(\mu_{\hat{\ell}}(\Pi_X), I_x)$

$\mu_{\hat{\ell}}(\Pi_X) \xrightarrow{\sim} I_x$

...mic rigidity

for inertia subgp

(2). (cf. [Cusp, Prop 2.3 (ii)])

$\forall$ divisor D of $\deg = 0$ on X s.t. $\mathrm{Supp}(D) \subset X(k)$,

the section $s_D : G_k \to \Pi_J$ is equal to (up to conj by Δ_X) the section det'd by the origin $O \in J(k)$

$\Longleftrightarrow$ D is principal

(cf. [NTs, Lem 4.14], [Naka, Claim (2.2)])

X/k

J^d $\deg = d$

$J = J^0$-torsor

$X \to J^1$

$P \mapsto \mathcal{O}(P)$

$\Pi_X \to \Pi_{J^1}$

$x \in X(k) \rightsquigarrow$

$s_x : G_k \to \Pi_X \to \Pi_{J^1}$

$\Pi_{J^1} \times \cdots \to \Pi_{J^1} \to \Pi_{J^d}$

div D $\deg = d$ s.t. $\mathrm{Supp}(D) \subset X(k)$

$\rightsquigarrow s_D : G_k \to \Pi_{J^d}$

$\bar{k} \supset \bar{k}_{NF} \supset \mathbb{Q}$

alg. closure

If $X_{\bar{k}}$ def'd / $\bar{k}_{NF}$, we say X is an <u>NF-curve</u>

$\bar{k} \subset \bar{k}(X)$

X: NF-curve, pts of $X(\bar{k})$ (resp. rat. fcts on $X_{\bar{k}}$, const. rat. fcts)

which descend to $\bar{k}_{NF}$

we call them <u>NF-points</u> (resp. <u>NF-rational functions</u>, <u>NF-constants</u>) on $X_{\bar{k}}$

Lem 3.16 ([AbsTopIII, Prop 1.8])

k : Kummer-faithful

$\emptyset \neq U \underset{\text{open}}{\subset} X \qquad S := X \setminus U$

Assume U : NF-curve ($\Rightarrow X$ is also NF curve)

$P_U \subset H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$: inverse image of $P'_U \subset \bigoplus_{x \in S} \mathbb{Z} \ (\subset \bigoplus_{x \in S} \hat{\mathbb{Z}})$

via $H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$

(ii). $\eta \in P_U$ is Kummer class of an NF-rat. fct.

$\Longleftrightarrow$ $\exists$ NF pts $x_1, x_2 \in U(k')$ (k'/k fin)

s.t. the restrictions $\eta|_{x_i} := s_{x_i}^*(\eta) \in H^1(G_{k'}, \mu_{\hat{\mathbb{Z}}}(\Pi_x))$

$\eta|_{x_1} = 0$, $\eta|_{x_2} \neq 0$ (section $G_{k'} \to \Pi_U$)

(additive expression) (multipl. $=1$, $\neq 1$)

η is in the subgp gen. by the elts char'zed by

(5). (cf. [Cusp, Prop 2.1 (ii)])

restrictions to I_x $(x \in S)$

$$0 \to H^1(\Pi_X, H^0(\prod_{x \in S} I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))) \to H^1(\Pi_U^{\text{cusp-cent}}, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} H^0(\Pi_X, H^1(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X)))$$

$H^1(\Pi_X, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \supseteq (k^\times)^\wedge$

$\hat{\mathbb{Z}}$

$\text{Hom}_{\Pi_X}(I_x, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

(2). Assume that $\exists$ non-const. NF-rat. fcts in $\Gamma(U, \mathcal{O}_U^\times)$

$\eta \in P_U \cap H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \supseteq (k^\times)^\wedge$

Kummer class of an NF-const. in $k^\times$

$(\Longleftarrow)$ $\exists$ a non-const NF-rat. fct $f \in \Gamma(U, \mathcal{O}_U^\times)$

& $\exists$ an NF-pt $x \in U(k')$ in k'/k fin.

s.t. $\kappa_U(f)|_x = \eta|_{k'}$ in $H^1(G_{k'}, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$(G_{k'}, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

$k' \to \Pi_U$

$\ne 1)$

NF

Th 3.17 (Mono-Anab. Reconstn. of NF-Portion [AbsTopIII, Th1.9])

k: sub-p-adic, X: str. Belyi type $\subset \overline{X}$

input $\mapsto \Delta_X \subset \Pi_X \twoheadrightarrow G_k \to 1$

We can (gp-thically) recon. $\overline{k}_{NF}(X)$, $\overline{k}_{NF}$ as following:

functorially

w.r.t. { open inj. hom. of ext'n of prof. gps
{ hom. of ext'n of prof. gp arising from a base change of the base field

Step 1 By Belyi cusp'zation (Th3.8)

(hom. of ... of ... base change
of the base field

Step 1 By Belyi cusp'zation (Th 3.8)

$\underbrace{}_{\text{gp th'cally}}$ the set of surj. $\{\Pi_U \twoheadrightarrow \Pi_X\}_U$

for open mult-NF-cusps $\phi \neq U \subset X$

& decomp. gps $D_x \subset \Pi_X$ of NF pts x

& inertia $I_x = D_x \cap \Delta_U$

Step 2 (Cyc. Rig. Iner.) (Prop 3.14)

$\underbrace{}_{\text{gp th'c}}$ $I_x \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(\Pi_X)$ $x \in X(k)$

step 1

Step 3

Step 1

write. $I_x \rightsquigarrow H^1(\Pi_v, \mu_{\hat{\ell}}(\Pi_x)) \to H^1(I_x, \mu_{\hat{\ell}}(\Pi_x))$

cycl. rig. Step 2 $\rightsquigarrow$ $1 \to \ker \to H^1(\Pi_v, \mu_{\hat{\ell}}(\Pi_x)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$

By the characterization of principal cuspidal div. (Lem 3.15 (5)) (Lem 3.15 (2) & decomp. gp in Step 1)

$\rightsquigarrow$ gp thic recon. $P_v \subset H^1(\Pi_v, \mu_{\hat{\ell}}(\Pi_v))$

Step 4

· By the char'ization of non const. NF-rat. fct's & NF-const. (Lem 3.16 (1), (2)) in P_v ← Step 3

$\overbrace{\text{gp th'c}}$ recon subgp (via Kummer map κ_v's)

$$\overline{k}_{NF}^{\times} \subset \overline{k}_{NF}(X)^{\times} \subset \varinjlim_{\text{ind. NF-cover of } \overline{X}\times k' \quad k'/k \text{ fin.}} H^1(\Pi_v, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$$

Step 1

...3.16 (1), (2)

$\leadsto$ Step 3

$(\Pi_X))$

$h'(h$
fin,

Step 5

(a) $\bar{h}_{NF}(X)^\times \leadsto$ Step 4

(b) ord_x's $\leadsto$ comp. at x of $\mathrm{hom}\ H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \bigoplus_{x \in S} \hat{\mathbb{Z}}$ (Step 3)

(c) $U_x \leadsto f(x) = 1 \leadsto$ next. no decomp. D_x

Uchida's lemma
Prop 3.12

gp th'c recon. additive str's on

$\bar{h}_{NF}^\times \cup \{0\}$, $\bar{h}_{NF}(X)^\times \cup \{0\}$

//

Prop 3.12 gp th'c recon. additive str's on

$\overline{k}_{NF}^{\times} \cup \{0\}$, $\overline{k}_{NF}(X)^{\times} \cup \{0\}$

OK.

Rem 3.17.1 k: NF, MLF $\leadsto$ input data only Π_X

Rem 3.17.2 $1 \to \Delta_{\eta_X} \to \Pi_{\eta_X} \to G_k \to 1$

gp. th. similar constr. (GC) no need of

k: Kummer-faithful $\exists$ in (Step 1). Belyi cusp'ization

$\leadsto$ $\overline{k}(X)$, $\overline{k}$

Rem 3.17.3 (slim of G_k for Kummer-faithful k) recon.

k: Kummer-faithful $\Rightarrow$ G_k: slim cf. [AbsTopIII, Thm 1.11, G proved], [GC, Lem 15.8]

VSP

Π mono

$\mathrm{Hom}(\Pi_1, \Pi_2)$ ...

mono Θ

$\mathbb{S}^{\Theta}$

$\mathbb{S}^{\pm}$, $\mathbb{S}^{\Theta}$

Θ $\mathbb{H}$ $\to$ $\mathbb{F}_\ell^{\times\pm}$

$\mathbb{F}_\ell^{\times} \times \{\pm 1\}$

$\pi_1^{\times}$

recon. gp in Step 1

Def. 3.18

$k/\mathbb{Q}_p$ fin. $\bar{k}$

$$\mu_{\mathbb{Q}/\mathbb{Z}}(G_k) := \varinjlim_{\substack{H \subset G_k \\ \text{open}}} (H^{ab})_{tor}, \quad \mu_{\hat{\mathbb{Z}}}(G_k) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\mathbb{Q}/\mathbb{Z}}(G_k))$$

Verlangerung

cyclotomes of G_k

Cor 3.19 (Mono-Anab. Recon. over MLF [AbsTopIII, Cor 1.10, Prop 3.2(ii), Rem 3.2.1])

$k/\mathbb{Q}_p$ fin. $\quad X$: hyperb. curve of str. Belyi type

prof. gp Π_X gp th'cally recon. the following

(1). the set of decomp. gps of all closed pts in X.

(2). $\bar{k}(X)$, $\bar{k}$ fields

(3). nat. isom. $\mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^{K}_{\hat{\mathbb{Z}}}(\Pi_X) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, K(\bar{k}^{\times}_{NF}))$

(Cyc. Rig. LCFT) cycl. rig. via LCFT, $K : \bar{k}^{\times}_{NF} \hookrightarrow \varinjlim_{U} H^1(\Pi_U, \mu_{\hat{\mathbb{Z}}}(\Pi_X))$

classical cycl. rig.

Def 3.18 $\quad k/\mathbb{Q}_p$ fin. $\quad \bar{k}$

$\ldots (\ldots^{ab}) \qquad \mu_{\hat{\mathbb{Z}}}(G_k) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mu_{\ldots}(G_k))$

Proof) (1). Cor 3.9

(2). Th 3.17 & Cor 2.4 $\rightsquigarrow \bar{h}_{NF}(X)$, $\bar{h}_{NF}$.

Prop 2.1 (7) $\overset{\text{gp thy}}{\underset{\mathrm{Hom}(\mathrm{ord},-)}{\rightsquigarrow}}$ $H^2(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \hat{\mathbb{Z}}$

cup prod.

$H^1(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \xrightarrow{\sim} \mathrm{Hom}(H^1(G_k, \hat{\mathbb{Z}}), \hat{\mathbb{Z}}) \cong G_k^{ab}$

$G_k^{ab} \twoheadrightarrow G_k^{ab}/\mathrm{Im}(I_k) \cong \hat{\mathbb{Z}}$

$\underset{\text{gp thy}}{\rightsquigarrow} H^1(G_k, \mu_{\hat{\mathbb{Z}}}(G_k)) \twoheadrightarrow \hat{\mathbb{Z}}$

$\mu_{\hat{\mathbb{Z}}}(G_k) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$ up to $\hat{\mathbb{Z}}^\times$-mult.

gp th'c recon k as completion of the field

$$\left(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \cap \bar{k}_{NF}^{\times}\right)^{\cup} \cup \{0\}$$

field str. on $\bar{k}_{NF}^{\times \cup} \cup \{0\}$

w.r.t. val. det'd by

the ordering of

$$\left(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \cap \bar{k}_{NF}^{\times}\right)^{\cup} \cup \{0\} \text{ gen. by}$$

$$\ker\left(H^1(G_k, \mu_{\hat{\mathbb{Z}}}(\Pi_X)) \to \hat{\mathbb{Z}}\right) \cap \bar{k}_{NF}^{\times}$$

indep. of the choice of $\mu_{\hat{\mathbb{Z}}}(G_k) \cong \mu_{\hat{\mathbb{Z}}}(\Pi_X)$

$\rightsquigarrow \bar{k}(X) := \bar{k} \otimes_{\bar{k}_{NF}} \bar{k}_{NF}(X)$.

Mono) (1). Cor 3.9

(3). $\mu^{k}_{\mathbb{Q}/\mathbb{Z}}(\Pi_X) := \mu^{k}_{\hat{\mathbb{Z}}}(\Pi_X) \otimes \mathbb{Q}/\mathbb{Z}$

Write $G^{ur} := \mathrm{Gal}(k^{ur}/k)$

Prop 2.1 (4a)

by the same way as Prop 2.1 (1),

$$H^2(G_k, \mu^{k}_{\mathbb{Q}/\mathbb{Z}}(\Pi_X)) \xrightarrow{\sim} H^2(G_k, k(\bar{k}^{\times})) \xleftarrow{\sim} H^2(G^{ur}, k((k^{ur})^{\times}))$$

$$\xrightarrow{\sim} H^2(G^{ur}, \mathbb{Z}) \xleftarrow{\sim} H^1(G^{ur}, \mathbb{Q}/\mathbb{Z})$$

$$= \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$$

Frob. elt.

$\xrightarrow{\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)}$ $H^2(G_k, \mu^{k}_{\hat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} \hat{\mathbb{Z}}$

$H^2(G_k, \mu_{\hat{\mathbb{Z}}}(G_k))^{\sim} \text{ in (2)} \rightsquigarrow \mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^{k}_{\hat{\mathbb{Z}}}(\Pi_X)$

(by imposing the compat. cond., we get) //

Step 5

$$\cong H^2(G^{ur}, \mathbb{Z}) \cong H^1(G^{ur}, \mathbb{Q}/\mathbb{Z})$$

$$= \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z}) \underset{\text{Frob. elt.}}{\cong} \mathbb{Q}/\mathbb{Z}$$

$$\overset{\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, -)}{\rightsquigarrow} \quad H^2(G_k, \mu^k_{\hat{\mathbb{Z}}}(\Pi_X)) \xrightarrow{\sim} \hat{\mathbb{Z}}$$

$$H^2(G_k, \mu_{\hat{\mathbb{Z}}}(G_k))^{\sim} \text{ in (2)} \quad \rightsquigarrow \mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu^k_{\hat{\mathbb{Z}}}(\Pi_X)$$

(by imposing the compat. cond. we get) //

<u>Rem 3.19.2</u> ([AbsTopIII, Prop 3.2, Prop 3.3])

$G_k \curvearrowright M$ top. monoid (resp. top. gp) $\cong \mathcal{O}^{\triangleright}_{\bar{k}}$ (resp. $\bar{k}^\times$)

compat. w/ G_k action

$$\mu_{\hat{\mathbb{Z}}}(M) := \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, M^\times)$$

$$\mu_{\mathbb{Q}/\mathbb{Z}}(M) := \mu_{\hat{\mathbb{Z}}}(M) \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}/\mathbb{Z}$$

$$M^{ur} := M^{\mathrm{Ker}(G_k \to G^{ur})}$$

$$M^{ur}/M^\times \cong \mathbb{N}$$

We can take the generator of $M^{ur}/M^\times$

(resp. gen. of $\underset{\cong \mathbb{Z}}{M^{ur}/M^\times}$ up to $\{\pm 1\}$)

$\rightsquigarrow H^2(G_k, \mu_{\mathbb{Q}/\mathbb{Z}}(M)) \xrightarrow{\sim} H^2(G_k, M^{gp}) \xleftarrow{\sim} H^2(G^{ur}, (M^{ur})^{gp})$

$\xrightarrow{\sim} H^2(G^{ur}, (M^{ur})^{gp}/(M^{ur})^{\times})$

$\xrightarrow{\sim} H^2(G^{ur}, \mathbb{Z})$

$(*)$

$\xleftarrow{\sim} H^1(G^{ur}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z})$

$\xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$

can. def'd (resp. well-def. up to $\{\pm 1\}$)

$\rightsquigarrow \mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(M)$ (Cyc. Rig. LCFT)

by the same Cor 3.19(3)

(resp. up to $\{\pm 1\}$) ways as "in"

$\mathcal{O}_{\bar{k}} \setminus \{0\}$

$\mathcal{O}^{\triangleright}_{\bar{k}}$ $\rightsquigarrow$ rig.

$\bar{k}^{\times}$ $\rightsquigarrow$ rig. up to $\{\pm 1\}$

$\mathcal{O}^{\times}_{\bar{k}}$ $\rightsquigarrow$ rig. up to $\hat{\mathbb{Z}}^{\times}$

= no rigidity

theta fct

mult. HA and

theta value

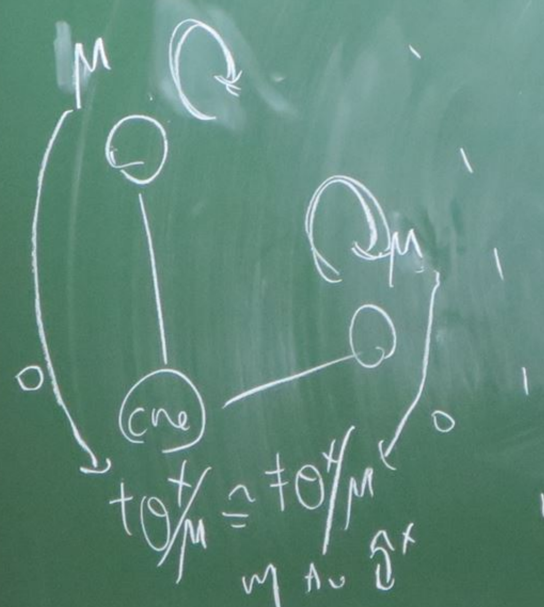
unimod.

log. + indet.

multirad.

§ 4. Archimedean Theory — Avoiding Specific Reference Model $\mathbb{C}$

ence Model $\mathbb{C}$



$\mathcal{O}_{\hbar}\setminus\{0\}$

$\mathcal{O}^{\triangleright}_{\hbar}$ ⤳ rig.

$\underline{\hbar}^{\times}$ ⤳ rig. up to $\{\pm 1\}$

$\mathcal{O}^{\times}_{\hbar}$ ⤳ rig. up to $\hat{\mathbb{Z}}^{\times}$ = no rigidity

$\xrightarrow{\sim} H^2(G^{ur}, \mathbb{Z})$ (*)

$\hookleftarrow H^1(G^{ur}, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(G^{ur}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$

can. def'd (resp. mult-def. up to $\{\pm 1\}$)

$\mu_{\hat{\mathbb{Z}}}(G_k) \xrightarrow{\sim} \mu_{\hat{\mathbb{Z}}}(M)$ (Cyc. Rig. LCFT)

by the same (in §1(3))

(resp. up to $\{\pm 1\}$) rigid as in

theta fct mult. HA and theta value unimod.

} log + inter multirad.

§4. Archimedean Theory — Avoiding Specific Reference Model $\mathbb{C}$

omit ([AbsTopIII, §2,4])

curve $/\mathbb{C}$ $X/\mathbb{C}$ $\mathbb{C}$ ⤳ $(\mathcal{O}^{\times}_{\hbar})^{\mu} = \hbar$

$X \begin{cases} X : \text{top.} \\ \mathbb{A}_X : U \mapsto \mathbb{A}_X(U) = \mathrm{Aut}^{hol}(U) \\ \quad U \subseteq X^{open}, \quad \mathbb{A}_X(U) \subseteq \mathrm{Aut}(U) \end{cases}$

Aut-hol. space

exterior — interior

$\mathbb{A}_X$

$\mathbb{C} \supseteq \mathcal{O}^{\triangleright}$, $\mathcal{O}^{\times}$ → $\mathbb{C}^{\times}$

X ⤳ ①

Kummer theory

$X \to Y$, $X \to Y$

$\mu(M \xrightarrow{\sim} M)$ "$+\mathbb{C}$" "$+\mathbb{C}$"

log $\mathcal{O}^{\times}_{\hbar} \subseteq \mathcal{O}^{\triangleright}_{\hbar} \subseteq \hbar$

§5. Log-Volumes and Log-Shells

§5.1 non-Arch. Places

$k/\mathbb{Q}_p$ fin. $\subset \bar{k}$

$$\tilde{k} := (\mathcal{O}^\times_{\bar{k}})^{pf} := \varinjlim \left(\mathcal{O}^\times_{\bar{k}} \xrightarrow{n} \mathcal{O}^\times_{\bar{k}} \to \cdots\right)$$

$$\left(\begin{matrix} \uparrow \\ \mathcal{O}^\times_{\bar{k}} \end{matrix}\right)$$

p-adic log: $\log_{\bar{k}} : \tilde{k} \xrightarrow{\sim} \bar{k}$

ring str.

field str.

input

$\mathcal{O}^\triangle_{\bar{k}}$

$\mathcal{O}^\times/\mu \cong \pm\mathcal{O}^\times/\mu$

X/k hyperbolic

$$\mathcal{O}_k^{\triangleright} \supset \mathcal{O}_k^{\times} \to \tilde{k} = (\mathcal{O}_{\bar{k}}^{\triangleright})^{gp} := (\mathcal{O}_{\bar{k}}^{\triangleright})^{gp} \cup \{0\} \leftarrow \mathcal{O}_{\bar{k}}^{\triangleright}$$

$\circlearrowleft \Pi_X$

Def 5.1

$$\mathcal{I}_k := \frac{1}{p^*} \mathrm{Im}\{\mathcal{O}_k^{\times} \to (\mathcal{O}_{\bar{k}}^{\times})^{pf} = \tilde{k}\} \quad (\mathcal{O}_{\bar{k}}^{\triangleright} \ \bar{k})^{\bar{k}}$$

$$(\mathcal{O}_{\bar{k}}^{\Pi_X} \subset)$$

$$p^* = \begin{cases} p & p > 2 \\ p^2 & p = 2 \end{cases}$$

log-shell of k, nahi.

$\mathcal{O}_{\hat{h}}^{\times \mu} \subseteq \mathcal{I}_h$ (common upper bound)

$\bigcup$

$\log \rightarrow \log(\mathcal{O}_h^{\times})$

$\boxplus$ $\log$ $\mathcal{O}^{\times \mu} \cong \mathcal{O}^{\times}$

$\boxtimes$ $\mathcal{O}^{\times \mu}$

stable

$\boxtimes$ - "ho" "ldy" + Θ - "lik" "compat." + $\log$-~~vol~~

$\boxplus$ - "ho" "ldy" + ~~...~~ + $\log$-vol $\longrightarrow \mathbb{R}_{\geq 0}$

$k \supset A \quad A \xrightarrow{\sim} \log(A)$

$\log\text{-vol}(A) = \log\text{-vol}(\log(A))$

$\text{vol}(\mathcal{O}_k) = 1$

$\text{vol}(\mathfrak{m}_k) = \frac{1}{q}$

(Ind3)

(Ind1?)

Θ NF, $\mathbb{F}_l^{\times \pm}$-symm.

$\log(\mu) = 0 \quad \mu$

$F^{\times} \cap \prod \mathcal{O}_v^{\times} = \mu$

non-interference

upper semi-compatibility

log-Kummer correspondence

$-\mathcal{O}^{\rhd}_{\underline{k}}$

$\cap$

$\underline{k}^{\sim}$

, arch.

k,

Prop 5.2 (Mono-Analytic Recon. of Log-Shell and Local Log-Volume in non-Arch. place)
[AbsTopIII, Prop 5.8 (i), (ii), (iii)]

G top. gp $\cong G_{\underline{k}}$

By the follow. algm, $G \rightsquigarrow$ gp th. log-shell "$\mathcal{I}_{\underline{k}}$" & (non-normalized) local log-volume fct "$\mu^{\log}_{\underline{k}}$"
in a functorial manner w.r.t. open hom. of top. gps.

Step 1 p, f_G, e_G, $\overline{k}^{\times}(G)$, $\mathcal{O}^{\rhd}_{\overline{k}}(G)$, $\mathcal{O}^{\times}_{\overline{k}}(G)$ by Prop 2.1 (1), (3b), (3c), (2a), (2c), (2d)

$p_G^{m_G}$ = the number of elts of $\overline{k}^{\times}(G)^G$ of p_G-power order.

$\boxtimes$-hol. lattice + $\boxtimes$-lab. compat. + ...

$\boxplus$-hol. latt. + ... + log-...

Step 2 $\quad \mathcal{I}(G) := \dfrac{1}{p_G^*} \mathrm{Im}\left(\mathcal{O}_{k}^{\times}(G)^{G} \longrightarrow k^{\sim}(G) := \mathcal{O}_{\bar{k}}^{\times}(G)^{pf} \right)$

$$p_G^* := \begin{cases} p_G & p_G > 2 \\ p_G^2 & p_G = 2 \end{cases} \qquad \mathbb{Q} \hookrightarrow \mathrm{End}(k^{\sim}(G))$$

Step 3 $\quad R_{\mathrm{nm}}(G) := \left(\bar{k}^{\times}(G) \,/\, \mathcal{O}_{\bar{k}}^{\times}(G) \right)^{\wedge}$

completion w.r.t. the order str. det'd by the image of $\mathcal{O}_{\bar{k}}^{\triangleright}(G)/\mathcal{O}_{\bar{k}}^{\times}(G)$

$\mathrm{End}(\mathbb{R}_{non}(G)) \cong \mathbb{R}$ $\quad$ $\mathbb{R}_{non}(G)$: $\mathbb{R}$-module

$\cup$

dist. elt $\mathbb{F}(G)$ $\leftarrow$ Frob. elt $\in \mathcal{O}^{\triangleright}_{k}(G)^{G} / \mathcal{O}^{\times}_{k}(G)^{G}$

$f_G \log k_G \in \mathbb{R}$ $\quad$ $\curvearrowright$ field $\quad$ $\mathbb{R}$-module

$\mathbb{F}(G) \in \mathbb{R}_{non}(G)$ $\quad$ input

Step 4 $\quad$ $\mathbb{M}(k^{\sim}(G)^{G})$: the set of open cpt subsets of the top. additive gp $k^{\sim}(G)^{G}$

recon. local log-vol fct $\mu^{\log}(G) : \mathbb{M}(k^{\sim}(G)^{G}) \longrightarrow \mathbb{R}_{non}(G)$

by the following properties

Step 3 $\quad \widetilde{I}(G) \ldots \frac{1}{\ldots} I \ldots (\mathcal{O}^{\times}(G)^{G} \ldots k^{\sim}(G) := \mathcal{O}^{\times}_{\ldots}(G)^{\mu f}$

(a) $A, B \in \mathcal{M}(\mathbb{R}^{?}(G)^{G})$ w/ $A \cap B = \phi$

(additivity)

$$\exp(\mu^{\log}(G)(A \cup B)) = \exp(\mu^{\log}(G)(A)) + \exp(\mu^{\log}(G)(B))$$

} field ith. in $R_{non}(G)$

(b) $A \in \mathcal{M}(\mathbb{R}^{?}(G)^{G})$, $a \in \mathbb{R}^{?}(G)^{G}$, $\mu^{\log}(G)(A+a) = \mu^{\log}(G)(A)$

(+-transl. inv.)

(c)

(normalization) $\mu^{\log}(G)(\mathcal{I}(G)) = \left(-1 - \frac{m_G}{f_G} + \epsilon_G e_G f_G\right)|\mathbb{F}(G)|$, $\epsilon_G = \begin{cases} 1 & p_G > 2 \\ 2 & p_G = 2 \end{cases}$

§5.2 Arch. ← [AbsTopIII §5]

§5.3 Globalize

$\mu^{\ell}(\theta) = 0$

$\mu^{\ell}(m) = -\log q$

$\frac{1}{(\ :\ )}$

(hindet)

<u>Prop 5.2</u> (Mono-Analytic R

[AbsTopIII,

G top. gp $\cong G_k$

By the follow.
alg'm, $G \underset{\text{gp th'c}}{\rightsquigarrow}$ log-shell "$\mathcal{I}_k$"

<u>Step 1</u> p, f_G, e_G, $\bar{k}$

$p_G^{m_G}$ = the

$h^n(G)^G$

additive gp

$\mathrm{rm}(G)$

$\sum_n \mu_n^{\log}$

can calcul deg of arith line bdl by this

g^{j^2}

1

ht $\sum_n t\mu_n^{\log}$

$\sum_n t\mu_n^{\log}$

$\to g(0)$

$g\,O \subset g^{j^2} O$

ht $\lesssim O$ + tindet

$= -\log q$

§6. Preliminaries on Tempered Fundamental Groups

$/\mathbb{C}$ $\quad \pi_1^{alg}$ $\quad$ alg. var. $\rightsquigarrow$ cpx mfd

$\qquad\qquad\quad \pi_1^{top} \longleftarrow$

$\qquad$ prof. $\quad$ disc.

$\qquad\qquad\quad$ alg. var. $\rightsquigarrow$ rigid var.

$/\mathbb{Q}_p$ $\quad \pi_1^{alg} \longleftarrow$

π_1^{top} $\quad \pi_1^{top}(\mathbb{P}^1_{\mathbb{C}_p} \setminus (N \text{ pts})) = \{1\}$ $\quad$ too small

no good $\quad \pi^{ét}$ $\quad \pi_1^{ét}(\mathbb{P}^1_{\mathbb{C}_p}) \twoheadrightarrow SL_2(\mathbb{Q}_p)$ $\quad$ too big

Gross-Hopkins

André $\rightsquigarrow$ tempered covering / fund. gp

$= -\log q$

$Y \to X$ étale covering of rigid analytic spaces

is <u>tempered</u> $\overset{\text{def}}{\Longleftrightarrow}$ $\exists$

$$\begin{array}{ccc} Z & \xrightarrow{\text{top.}} & T \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

$\leftarrow$ possibly infinite degree

dig. of fin. étale cov'g

$\rightsquigarrow \pi_1^{\text{temp}}$

e.g. $\pi_1^{\text{temp}}(\mathbb{P}^1_{\mathbb{C}_p} \setminus \{0, \infty\}) = \hat{\mathbb{Z}}$

E ell. cu/$\mathbb{C}_p$ $\quad \pi_1^{\text{temp}}(E) \cong \begin{cases} \hat{\mathbb{Z}} \times \hat{\mathbb{Z}} & |j|_p \le 1 \\ \mathbb{Z} \times \hat{\mathbb{Z}} & |j|_p > 1 \end{cases}$

π_1^{top} : neither discrete, nor profinite, nor locally compact, but pro-discrete

§6. Preliminaries on Tempered Fundamental Groups

Def. 6.1 ([SemiAnbd, Def. 3.1 (i), Def. 3.4])

(1). Anp. gp $\Pi \overset{?}{\cong} \varprojlim_{\text{inj's}}$ (countable disc. top. gp)

tempered gp

(2). Π : tempered gp. Π : temp-slim $\overset{\text{def}}{\Longleftrightarrow}$ $Z_\Pi(H) = \{1\}$ $\forall$ open $H \subset \Pi$

Cor 2.4 ⇒ Lem 6.2

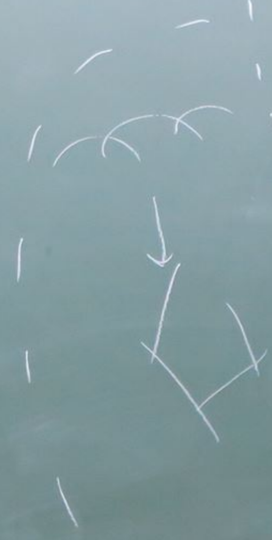
X: hyperb$/k$ $\quad \Pi_X^{tmp} \rightsquigarrow \Delta_X^{tmp}$

gp th'c

$\left(☺ \quad (\Pi^{tmp})^\wedge = \Pi \right)$

$F \supset K / \mathbb{Q}_p$ fin.

$\overline{F} \supset \overline{k}$ vs. field



[IUTch II]

HA and

Π^{tmp} / k_v

labeling by global data

~~temp~~ $\hat{\Pi} / K$

mult. vs tmp. cj. $\in \Pi^{tmp}$

Def. 6.3 (1). $\overline{X}$: pt'd $\overset{\text{nodes}}{\text{stable}}$ curve $/ \overline{k}$ w/ marked pts D

$X := \overline{X} \setminus D$

$\rightsquigarrow$ dual semi-graph (resp. dual graph) $\mathbb{G}_X$

$X_v \rightsquigarrow v$ vertices : irred. comp's of X

$v_e \rightsquigarrow e$ edges : nodes $\longleftarrow$ $\overset{\text{closed}}{\text{edge}}$ joint vertices corresp. to the irred. comp.'s on which

cusps $\longleftarrow$ open edge just to the vertex corresp. to the irred. comp. on which the node lies / the cusp lies

(resp. edges : nodes $\longleftarrow$ . . .)

(2). (cf. [Anbd, Appendix])

$X \rightsquigarrow \mathcal{G}_X$: semi-graph of Galois cat.

$\Sigma \subset \mathfrak{Primes}$

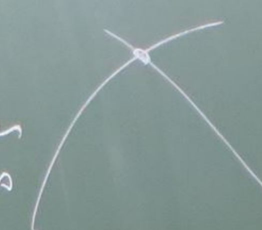
underlying graph $= \mathbb{G}_X$

(pro-Σ) e.g. $\Sigma = \{l\}$

$X' := X \setminus \{\text{nodes}\}$, $\mathbb{G}_X \ni v$ vertex

$\mathcal{G}_v$: the Gal. cat. of the fin. étale cov's of X'_v

$(X'_v := X_v \times_X X')$ which are tamely ram. along the nodes & cusps



$\subset \beta$ ius

Σ) e.g. $\Sigma = \{\ell\}$

étale cov. gr

X'_n

$v_e \overset{\text{branches}}{\rightsquigarrow} v_e(1), v_e(2)$; schematic

$X'_{v_e(i)}$: intersection of the completion of the branch $v_e(i)$ at the node v_e w/ X'

$(i = 1, 2)$

non-can. $\mathrm{Spec}\, \bar{k}((t))$

fix $\bar{k}$-isom $X'_{v_e(1)} \cong X'_{v_e(2)}$ ← identity

$\rightsquigarrow X'_e$

$\mathcal{G}_e$: the Gal. gp. of the fin. et. (pro-Σ) cov.'s of X'_e which are tamely ram. along the node

X' : sch. thic

open edge e_x $\quad$ X'_x : sch. theoretic intersection of the completion of $\overline{X}$ at x w/ X'

x : cusp

non-can. $\mathrm{Spec}\, \overline{k}((t))$ (non-Σ)

$\mathcal{G}_{e_x}$: Gal. cat. of the fin. ét. cov.'s of X'_x which are tame ram. along the cusp.

$$\begin{array}{ccc} & e & \\ \swarrow & & \searrow \\ v_1 & & v_2 \end{array}$$

pull-backs ⇝ natural functors

$$\begin{tikzcd} \mathcal{G}_{v_1} \arrow[r] & \mathcal{G}_e \\ \mathcal{G}_{v_2} \arrow[ur] & \end{tikzcd}$$

open edge $e \to v$

$$\mathcal{G}_v \to \mathcal{G}_e$$

$$\mathcal{G}_X := \{\mathcal{G}_v ; \mathcal{G}_e ; \mathcal{G}_v \to \mathcal{G}_e\}$$

$v_e \rightsquigarrow$ branches $v_e(1), v_e(2)$; scheme-theoretic intersection of the completion of the branch $v_e(1)$ at the node v_e

X' ;

Isom. in the cat. of
fund cat. cyclmod. of
com. obj's of $\mathcal{G}_e$

(3) [cf. [SemiAnbd, Def 2.1]]

<u>Ficomm</u> $\mathcal{G} = \{\mathcal{G}_v ; \mathcal{G}_e ; \mathcal{G}_v \to \mathcal{G}_e\} / G$

$\rightsquigarrow$ cat $\mathcal{B}(\mathcal{G})$

<u>Obj</u> data $\{S_v, \phi_e\}_{v,e}$

vertices edges

$\pi_1(X, Y)/(XY)$

$x^1 \mapsto x^e$, $y^1 \mapsto y^e$

$\pi_1(X, Y)/(X'Y')$

$S_v \in \mathrm{Ob}(\mathcal{G}_v)$

$\phi_e : e(1)^* S_{v_1} \xrightarrow{\sim} e(2)^* S_{v_2}$

branches of e

$e(i)^* : \mathcal{G}_{v_i} \to \mathcal{G}_e$

<u>Hom</u> evident manner.

$\mathcal{B}(\mathcal{G})$: Galois cat. $\mathcal{G} = \mathcal{G}_X$ in (2) $\rightsquigarrow \pi_1(\mathcal{B}(\mathcal{G}))$ admissible fund. gp

(4). (cf. [SemiAnbd, before Def 3.9 & Def 3.9])

$\mathcal{G}$: semi. $\mathcal{G} = \{\mathcal{G}_v ; \mathcal{G}_e ; \mathcal{G}_v \to \mathcal{G}_e\}$

$\rightsquigarrow$ cat $\mathcal{B}^{\mathrm{cov}}(\mathcal{G})$: <u>Obj</u> data $\{S_v, \phi_e\}_{v,e}$ $\amalg$

<u>Mor</u> evident manner.

S_v : finite countable coproduct of conn. obj's of $\mathcal{G}_v$

$\phi_e : e(1)^* S_{v_1} \xrightarrow{\sim} e(2)^* S_{v_2}$ isom. in the cat. of fin'l cat. coprod. of conn. obj's of $\mathcal{G}_e$

($\mathcal{G}$: countable semi-graph $\rightsquigarrow \mathcal{B}^{\mathrm{cov}}(\mathcal{G})$)

(3) [cf. [SemiAnbd, Def 2.1])

$\mathcal{G} = \{\mathcal{G}_v ; \mathcal{G}_e ; \ldots$

$\pi_1(X, \bar{x})/(X, \bar{x})$

along the vertices & cusps

$(\mathcal{B}(G) \subset (\mathcal{B}^{\mathrm{spl}}(g) \subset \mathcal{B}^{\mathrm{cov}}(g))$

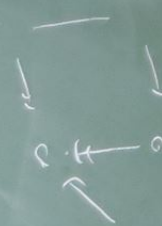
full subcat.

obj. $\{S_v, \phi_e\}_{v,e}$ s.t.

$\exists \{S'_v, \phi'_e\}_{v,e} \in \mathrm{Obj}(\mathcal{B}(g))$

$\forall$ vertex or edge c restriction of $\{S'_v, \phi'_e\}$ to g_c splits the restriction of $\{S_v, \phi_e\}$ to g_c

i.e. the finite product of S'_v (resp. ϕ'_e) w/ S_v (resp. ϕ_e) in the term of obj. (resp. id. morph.) of the obj. (resp. g_e) in the cat. of fin. ℓ (countable coprod. of connected obj.) of G_v



& cusps

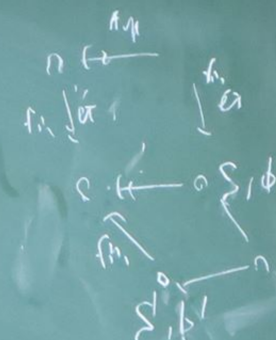
$\mathcal{G}_e$: the Gal. cat. of the [illegible]
which are [illegible]

$\mathcal{B}^{(cov)}(\mathcal{G})$

[illegible] subcat.

[illegible]: $\{S_v, \phi_e\}_{v,e}$ s.t.

$\exists \{S'_v, \phi'_e\}_{v,e} \in \mathrm{Obj}(\mathcal{B}(\mathcal{G}))$



$\forall$ vertex or edge c restriction of $\{S'_v, \phi'_e\}$ to $\mathcal{G}_c$ splits the restriction of $\{S_v, \phi_e\}$ to $\mathcal{G}_c$

i.e. the fiber product of S'_v (resp. ϕ'_e) w/ S_v (resp. ϕ_e) over the term. obj. (resp. id. morph. of term. obj.) in the cat. of [illegible] countable coprod. of conn. obj. of $\mathcal{G}_v$ (resp. $\mathcal{G}_e$)

is isom. to the coprod. of [illegible]

Rem 6.3.1 (cf. [SemiAnbd, Ex. 3[illegible]

X : smooth log. curve [illegible]

special fiber of the [illegible]

$\rightsquigarrow \mathcal{G}$: [illegible]

(2). (cf. [AbsAnab, Appendix])

$X \rightsquigarrow \mathcal{G}_X$: semi-graph of Galois cat.

underly graph $= \mathbb{G}_X$

$X' := X \setminus \{\text{nodes}\}$, $\mathbb{G}_X \ni v$ vertex

$\mathcal{G}_v$: the Gal. cat.

$\left(X'_v := X_v \times_X X' \right)$ which ... tamely ... along the ... & ...

...mod. et

$\mathcal{G}_v$

...$_2$

...mod. of

...of $\mathcal{G}_v$

Σ C β rias

…o-Σ) e.g. Σ = {l}

…in. étale cover

…of X'_n

is isom. to the coprod. of a countable number of copies of $\hat{\mathbb{Z}}'_1$ (resp. $\mathbb{F}'_p$)

<u>Rem. 6.3.1</u> (cf. [SemiAnbd, Ex. 3.10])

X : smooth curve $/\overline{\mathbb{F}}_l$ w/ marked pts + log

special fiber of the stable model of X

$\leadsto \mathcal{G}$: semi-graph of C.D. cat's

$\leadsto \pi_1(\mathcal{B}^{\mathrm{temp}}(\mathcal{G}))$

$\Delta_X^{\mathrm{Temp}} := \pi_1^{\mathrm{temp}}(X)$

$\cup N_i$) .....)

exhaustive seq. of open char. subgp of fin. index.

$\updownarrow$

$X_i \to X$

$\leadsto \mathcal{G}_i \curvearrowleft \Delta_X^{\mathrm{temp}}/N_i$

faithful

$\rightsquigarrow \cdots \leftarrow \Pi^{top}(g_i) \leftarrow \cdots \leftarrow \Pi^{top}(g)$

compatible w/ actions of Δ_X^{top}/N_i

$\rightsquigarrow \Delta_X^{top} \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(\Pi^{top}(g_i)) \overset{out}{\rtimes} (\Delta_X^{top}/N_i)$

$\twoheadrightarrow \cdots \twoheadrightarrow \pi_1^{top}(g)$

$\rightsquigarrow \Delta_X^{top} \cong \varprojlim_i \left(\pi_1^{top}(\Pi^{top}(g_i)) \overset{out}{\rtimes} (\Delta_X^{top}/N_i) \right)$

$*$

is isom. to the coprod. of a countable number of copies of $\hat{\mathbb{Z}}$ (resp. $\mathbb{Z}_l$)

$\rightsquigarrow \pi_1(\Pi^{top}(g))$

Prop 6.4 (cf. [SemiAnbd, Prop 2.6, Cor 2.7, Prop 3.6])

$\pi_1(\mathcal{B}^{\text{temp}}(\mathcal{G}_{\cdot}))$: temp-ali

$\underset{\otimes}{\rightsquigarrow}$ Δ_X^{temp} : temp-ali $\rightsquigarrow$ Π_X^{temp} : temp-ali

Frobenioid

Fr'd $\boxed{\varepsilon\tilde{\iota}\delta o\varsigma}$

ἰδη

ἄνθρωπος

$\Pi_X^{\text{temp}}[\mu_N]$

$\cup$

μ_N

Prop 6.5 (cf. [SemiAnbd, Th 3.7])

($\pi_1^{\mathrm{top}}(g)$:= $\pi_1(\mathbb{B}^{\mathrm{top}}(g))$)

(1). max. cpt subgps of $\pi_1^{\mathrm{top}}(g)$ $\overset{1:1}{\longleftrightarrow}$ verticial subgps of $\pi_1^{\mathrm{top}}(g)$ $\overset{1:1}{\longleftrightarrow}$ vertex

up to conj.

image of $\pi_1(g_v)$ for v: vertex

(2). $\forall$ edge-like subgp in $\pi_1^{\mathrm{top}}(g)$ $\overset{1:1}{\longleftrightarrow}$ edge

image of $\pi_1(g_e)$ for e: closed edge

is contained in precisely 2 verticial subgps

((3). $\exists$ gp th'c char. for cusp. decp. gp in $\pi_1^{\mathrm{top}}(g)$

$\rightsquigarrow$ In particular gp th'c can recover $\mathbb{G}_X$)

⊗ & pro-Σ $\Sigma \not\ni p$ $\Rightarrow$ $\pi_1^{\mathrm{adm}}(\underline{\text{special}}) \cong \pi_1^{\Sigma}(X_{\bar{K}})$ $\rightsquigarrow$ $\pi_1(X_{\bar{K}})$ $\underset{\text{gp th'c}}{\rightsquigarrow}$ graph of special fiber

Frobenioid

$\Sigma = \{l\}$

$l \neq p$

Thm 6.6 (Prof. Conj. vs Top. Conj [IUTchI, Prop 2.4, Cor 2.5])

$$\gamma \in \hat{\Pi}_X \text{ s.t. } \gamma \cap \hat{\gamma}^{-1} \subset \Delta^{top}_X \Rightarrow \gamma \in \Pi^{top}_X$$

$\{l\}^{\wedge}$ cpt sub $\subset \Delta^{top}_X$

Cor 6.7 $\quad \Pi^{top}_X \underset{\text{gp th'ic}}{\rightsquigarrow}$ comp. dom. gps

Π^{top}_X s.t. nice in G_k open

$\Uparrow$

Cor 2.9, Rem 2.9.1, Thm 6.6 $\leftarrow$ prof. cpl. & Thm 6.6